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Reliability of uncertain dynamical systems with multiple design points

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Abstract

Asymptotic approximations and importance sampling methods are presented for evaluating a class of probability integrals with multiple design points that may arise in the calculation of the reliability of uncertain dynamical systems. An approximation based on asymptotics is used as a first step to provide a computationally efficient estimate of the probability integral. The importance sampling method utilizes information of the integrand at the design points to substantially accelerate the convergence of available importance sampling methods that use information from one design point only. Implementation issues related to the choice of importance sampling density and sample generation for reducing the variance of the estimate are addressed. The computational efficiency and improved accuracy of the proposed methods is demonstrated by investigating the reliability of structures equipped with a tuned mass damper for which multiple design points are shown to contribute significantly to the value of the reliability integral.

Keywords: Structural reliability; Asymptotic approximation; Importance sampling; Dynamical systems; Tuned mass dampers

1. Introduction

Structural reliability analyses involve the development of accurate and efficient methods for computing multi-dimensional probability integrals. Two classes of methods are widely used to compute structural reliability or its complement, the probability of failure. The first class consists of first and second-order reliability methods (FORM and SORM) [1–5] which have been developed to provide economical computational tools for approximating structural reliability. The second class consists of Monte-Carlo simulation methods [6], including importance sampling methods, which can improve the reliability estimate to any desirable degree of accuracy at the expense of more computational effort [7–15].

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FORM and SORM methods are applied to the classical reliability integral

$$I = \int_{\mathbb{F}} p(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{1}$$

where $\mathbb{F}$ is the failure domain defined by the limit state function $g(\boldsymbol{\theta})$ as $\mathbb{F} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} : g(\boldsymbol{\theta}) \leqslant 0\}$ and $p(\boldsymbol{\theta})$ is the probability density function of $\boldsymbol{\theta}$. These methods involve the computation of the design point, which is defined as the point in the failure domain that is closest to the origin in the standard Normal space after transformation of the original random variables [1]. Such a point has the highest probability density among all points in the failure domain and is usually found as the solution of a constrained optimization problem [16].

Another important class of multi-dimensional probability integrals, arising in the formulation of the reliability analysis of uncertain dynamical systems subjected to stochastic excitations, is of the form [15,17,18]

$$I = \int_{\boldsymbol{\Theta}} F(\boldsymbol{\theta}) p(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{2}$$

where $F(\boldsymbol{\theta})$ and $p(\boldsymbol{\theta})$ are positive smooth functions of $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ with $\boldsymbol{\Theta}$ being a subset of $\mathbb{R}^n$, and $F(\boldsymbol{\theta})$ represents the conditional probability of failure for the system given the uncertain parameters $\boldsymbol{\theta}$. Design points for this integral are defined equivalently as the points which maximize the integrand either locally or globally. In this case, these 'local' or 'global' design points are usually found as a solution to an unconstrained optimization problem.

The main contribution to the reliability integral in general comes from the neighborhood of design points. When multiple design points exist, available optimization algorithms may converge to a local design point and thus erroneously neglect the main contribution to the value of the reliability integral from the global design point(s). Moreover, even if a global design point is obtained, there are cases for which the contribution from other local or global design points may be significant. Importance sampling strategies for time-invariant problems with reliability integrals of the form (1) having multiple design points have been addressed [7,11,14,19]. However, implementation issues and methodologies for finding and treating multiple design points have not been fully explored.

The focus of this work is the estimation of reliability integrals of the form (2) for uncertain dynamical systems where multiple design points exist. An approximation based on asymptotics which utilizes the information from multiple design points is studied. An efficient method for finding all the design points within a specified domain is also presented. In addition, an importance sampling method using information about the design points is proposed to improve the accuracy of the asymptotic estimate to any desirable degree at the expense of more computation. Implementation issues related to choice of importance sampling density and sample generation are addressed. The reliability of multi-story buildings with one or more tuned mass dampers is one example for which multiple design points are encountered and the contribution to the value of the reliability integral from more than one design point is significant. Numerical results corresponding to a single-degree-of-freedom structure and a 10-story shear building equipped with a tuned mass damper (TMD), each with several uncertain parameters, are presented to demonstrate the computational efficiency and improved accuracy of the proposed methods for multiple design points.

2. Asymptotic approximation

Motivated by the work of Breitung [5] on asymptotic approximations of reliability integral (1), Papadimitriou et al. [15] have derived an asymptotic approximation for integral (2). The idea is to expand the logarithm of the integrand about the "design points" that correspond to the maximum of the integrand and then make use of Laplace's method on the resulting integral [20, 21]. In the case of multiple design points, designated by $\boldsymbol{\theta}_1^*, \ldots, \boldsymbol{\theta}_M^*$, the asymptotic approximation is given by summing the contributions from the design points as

$$I \approx \hat{I} = \sum_{i=1}^{M} \hat{I}_i \tag{3}$$

where $\hat{I}_i$, $i = 1, \ldots, M$, is the "asymptotic contribution" to the reliability integral from the design point $\boldsymbol{\theta}_i^*$, given by

$$\hat{I}_i = (2\pi)^{n/2} \frac{F(\boldsymbol{\theta}_i^*)p(\boldsymbol{\theta}_i^*)}{\sqrt{|\mathbf{H}(\boldsymbol{\theta}_i^*)|}} \tag{4}$$

and $|\mathbf{H}(\boldsymbol{\theta})|$ is the determinant of the Hessian matrix $\mathbf{H}(\boldsymbol{\theta})$ of $-\ln[F(\boldsymbol{\theta})p(\boldsymbol{\theta})]$.

The quality of the approximation (3) with (4) depends on the decay of the function $F(\boldsymbol{\theta})p(\boldsymbol{\theta})$ in the neighborhood of the design points, as well as the distance between the different design points. In fact, the error in the approximation for a single design point $\boldsymbol{\theta}^*$ can be quantified as follows. First, define the fractional error in the integrand of (2) by:

$$\varepsilon(\boldsymbol{\theta}) = \frac{h(\boldsymbol{\theta}) - \hat{h}(\boldsymbol{\theta})}{\hat{h}(\boldsymbol{\theta})} \tag{5}$$

where

$$h(\boldsymbol{\theta}) = F(\boldsymbol{\theta})p(\boldsymbol{\theta}) \tag{6}$$

$$\hat{h}(\boldsymbol{\theta}) = h(\boldsymbol{\theta}^*) \exp\left[-\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T \mathbf{H}(\boldsymbol{\theta}^*)(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\right] \tag{7}$$

Note that the approximation $\hat{I}$ of (2) comes from integrating the local Normal approximation, $\hat{h}(\boldsymbol{\theta})$, of $h(\boldsymbol{\theta})$. If $\varepsilon(\boldsymbol{\theta})$ satisfies the condition:

$$\left|\varepsilon(\boldsymbol{\theta})\right| \leqslant K\lambda \left\|\boldsymbol{\theta}-\boldsymbol{\theta}^*\right\|^3, \text{ as } \lambda \rightarrow \infty, \tag{8}$$

for some positive constant K, then by applying Laplace's method, it can be shown that the fractional error in the integral (2) is given by:

$$\frac{I-\hat{I}}{\hat{I}} = O\left(\frac{1}{\sqrt{\lambda}}\right) \rightarrow 0, \text{ as } \lambda \rightarrow \infty, \tag{9}$$

where λ is the smallest eigenvalue of $\mathbf{H}(\boldsymbol{\theta}^*)$. Note that all the derivatives of $\varepsilon(\boldsymbol{\theta})$ up to, and including, the second derivatives are zero at $\boldsymbol{\theta}^*$. Therefore, if $\varepsilon(\boldsymbol{\theta})$ is sufficiently smooth, the bound in (8) is reasonable. This argument can be extended to multiple design points to also get an asymptotic result as $\lambda = \min_i \lambda_i \rightarrow \infty$, where λ_i is the smallest eigenvalue of $\mathbf{H}(\boldsymbol{\theta}_i^*)$. Based on this asymptotic result, for finite λ the result (3) with (4) is taken as an approximation for reliability integral (2).

Note that the approximation (3) and (4) can be applied directly to the integral (2) or it can be applied to the integral resulting from transforming the original variables $\boldsymbol{\theta}$ to independent and standard Normally distributed variables. While this transformation can always be done in principle through the Rosenblatt transformation, in many cases the transformation cannot be performed analytically and must be done numerically, which greatly increases the computational requirements. However, even for those cases that it is simple to transform the integral in the standard Normal space, it is a matter of preference as to which space the approximation should be applied to, since, depending on the application, the approximation in the transformed standard Normal space may give less accurate results than the one obtained in the original space.

The computationally most expensive operation in the asymptotic method is the search for the design points $\boldsymbol{\theta}_i^*$. In some practical applications, only one local maximum exists inside the region $\boldsymbol{\Theta}$, and so it can be readily obtained using a local maximization method such as the modified-Newton method. It should be noted, however, that when a good initial guess is not available, the modified-Newton method may not converge. In this case, a homotopy method can be used which provides a robust way to find at least one stationary point of the objective function [22]. In the case of multiple maxima, more sophisticated optimization methods are required for finding all local maxima.

A heuristic and robust method for finding multiple design points for reliability integrals of the form (1) has recently been developed [19]. The main idea of the method is to impose a 'barrier' around known design points by modifying the limit state function. Subsequent optimization using local optimization algorithms is then more likely to converge to new design points. However, the method is developed specifically for integral (1) in which a constrained global optimization is involved and cannot be applied directly to find multiple design points for integral (2).

Relaxation techniques [22] have been developed for reliably obtaining multiple maxima points $\boldsymbol{\theta}_i^*$ for unconstrained optimization problems arising in the approximation (4) of reliability integral (2). Once a stationary point is found using a local optimization method, the relaxation method is applied to find other stationary points as follows. Starting from a known stationary point, a trajectory is followed along which the stationarity condition with respect to one of the coordinates is relaxed while the rest remain enforced. A new stationary point is found when the relaxed condition is satisfied again along the trajectory, whose type (minimum, maximum or saddle point) can be checked using the Hessian matrix of the original objective function. Each of the n stationarity conditions are relaxed in turn to produce a network of n trajectories from each stationary point. By systematic branching of trajectories in this way from each stationary point as it is found, all of the stationary points in a specified domain of interest may be found.

The multiple design points in the numerical example have been found using the relaxation scheme. It is noted, however, that the asymptotic approximation and importance sampling methodology presented in this work do not depend on the search algorithm used and can be applied once the design points are found.

3. Importance sampling

In the importance sampling procedure, simulations are applied to the integral

$$I = \int_{\Theta} \frac{F(\boldsymbol{\theta})p(\boldsymbol{\theta})}{f(\boldsymbol{\theta})} f(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{10}$$

where $f(\boldsymbol{\theta})$ is the importance sampling density chosen so that most of the samples $\boldsymbol{\theta}^{(k)}$, $k = 1, \ldots, N$, are generated in the region or regions that contribute significantly to the integral. The estimate of I is given by the sample mean $\tilde{I}_N$ of $\kappa = Fp/f$:

$$\tilde{I}_N = \frac{1}{N} \sum_{k=1}^{N} \kappa(\boldsymbol{\theta}^{(k)}) \tag{11}$$

For large N the variance $\mathrm{Var}\left[\tilde{I}_N\right]$ of $\tilde{I}_N$ is estimated by

$$\tilde{\sigma}_N^2 = \mathrm{Var}\left[\tilde{I}_N\right] = \frac{1}{N} \sum_{k=1}^{N} [\kappa(\boldsymbol{\theta}^{(k)}) - \tilde{I}_N]^2 \tag{12}$$

Since the main contribution to the integral comes from the domains in the neighborhood of the design points $\boldsymbol{\theta}_1^*, \ldots, \boldsymbol{\theta}_M^*$, it is reasonable to choose $f(\boldsymbol{\theta})$ to have significant values at these design points. In the case of a single design point $\boldsymbol{\theta}^*$, the importance sampling distribution was chosen to have most probable value at the design point $\boldsymbol{\theta}^*$ [15]. Generalizing this idea to the case of multiple design points [7,11,14], the sampling distribution $f(\boldsymbol{\theta})$ is chosen to be of the form

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{M} w_i G_i(\boldsymbol{\theta}) \tag{13}$$

where $G_i(\boldsymbol{\theta})$, $i = 1, \ldots, M$, are specified probability density functions with most probable values equal to the design point $\boldsymbol{\theta}_i^*$, and the w_i are the corresponding weights associated with the distribution, satisfying $0 \leqslant w_i \leqslant 1$, $i = 1, \ldots, M$, and $\sum_{i=1}^{M} w_i = 1$.

The choice of the distributions G_i and the weights w_i are critical factors affecting the efficiency of the importance sampling procedure. It can be shown that choosing $f(\boldsymbol{\theta})$ to have the same tail behavior as $p(\boldsymbol{\theta})$ guarantees that the variance of $\kappa = Fp/f$ is finite if $F(\boldsymbol{\theta})$ has finite variance under $p(\boldsymbol{\theta})$ [7,14,15]. In particular, for a Normal distribution $p(\boldsymbol{\theta})$ with covariance matrix $\mathbf{C}$, a finite variance is guaranteed if the choice for the covariance matrix $\mathbf{C}_i$ of the Normal distribution G_i is such that the matrix $\mathbf{C}_i - \mathbf{C}$ is positive semidefinite. The choice that may accelerate the convergence of the importance sampling scheme is for $\mathbf{C}_i$ to be the inverse of the Hessian $\mathbf{H}(\boldsymbol{\theta}_i^*)$. However, this choice can only be made in the cases for which the matrix $\mathbf{H}^{-1}(\boldsymbol{\theta}^*) - \mathbf{C}$ is positive semi-definite so that it yields a finite variance. In all other cases, it is reasonable to let $\mathbf{C}_i = \lambda\ \mathbf{H}^{-1}(\boldsymbol{\theta}^*)$ and choose $\lambda (\lambda > 1)$ such that $\mathbf{C}_i - \mathbf{C}$ is positive semi-definite. An alternative choice which always satisfies the semi-definiteness of $\mathbf{C}_i - \mathbf{C}$ is $\mathbf{C}_i = \mathbf{C}$, the covariance matrix of the original Normal distribution. This choice provides computational advantages over the alternatives when $\mathbf{C}$ is a diagonal matrix.

If $p(\boldsymbol{\theta})$ is not a Normal distribution, there are several ways of applying the importance sampling technique which will guarantee a finite sample variance. One way is to map the original set of variables $\boldsymbol{\theta}$ into a new set of independent Normal variables and apply importance sampling to the transformed integral, as just described. Another way is to appropriately choose $f(\boldsymbol{\theta})$ in the original parameter space depending on the distribution $p(\boldsymbol{\theta})$. One such choice for an independent lognormal variable is given in [15] and it is used in the importance sampling estimate of the 2-DOF system considered in the applications section.

The generation of samples for random variables $\boldsymbol{\theta}$ with joint probability density function $f(\boldsymbol{\theta})$ given by (13) could be carried out as follows [7]. To generate the kth sample, $\boldsymbol{\theta}^{(k)}$, $k = 1, \ldots, N$, a discrete random variable u having $\{1, \ldots, M\}$ as its state-space with corresponding probabilities $\{w_1, \ldots, w_M\}$ is simulated first. If $u = i, \boldsymbol{\theta}^{(k)}$ is generated from G_i. The number of samples N_i generated from G_i is a Binomial random variable which has mean and variance equal to $w_i N$ and $w_i(1 - w_i)N$, respectively, and therefore, on average, the number of samples generated around the ith design point is proportional to the associated weight, w_i. The variance σ^2 of $\kappa = Fp/f$ is given by

$$\sigma^2 = \int_\Theta \kappa(\boldsymbol{\theta})^2 f(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} - I^2 \tag{14}$$

Substituting $f(\boldsymbol{\theta}) = \sum_{i=1}^{M} w_i G_i(\boldsymbol{\theta})$ and $I = \sum_{i=1}^{M} w_i I_i$, the variance of the importance sampling estimate takes the form

$$\mathrm{Var}\big[\tilde{I}_N\big] = \frac{\sigma^2}{N} = \frac{1}{N}\left[\bar{\sigma}^2 + \sum_{i=1}^{M} w_i(\sigma_i - \bar{\sigma})^2 + \sum_{i=1}^{M} w_i(I_i - I)^2\right] \tag{15}$$

where

$$\sigma_i^2 = \int_\Theta \kappa(\boldsymbol{\theta})^2 G_i(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} - I_i^2 \tag{16}$$

and

$$I_i = \int_{\Theta} \kappa(\boldsymbol{\theta}) G_i(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{17}$$

are, respectively, the variance and mean of $\kappa(\boldsymbol{\theta})$ under the sampling distribution $G_i(\boldsymbol{\theta})$, and $\bar{\sigma} = \sum_{i=1}^{M} w_i \sigma_i$ is the weighted average of the standard deviations σ_i.

Alternatively, the estimate for I can be carried out by independently computing the importance sampling estimate $\tilde{I}_{i,N_i}$ for the integral I_i given in (17) with N_i now being a fixed number. Substituting (13) into (10), the resulting estimate of I, denoted by $\tilde{I}_N^*$, is then given by

$$\tilde{I}_N^* = \sum_{i=1}^{M} w_i \tilde{I}_{i,N_i} \tag{18}$$

Note that the number of samples used in estimating I_i is fixed, whereas in the previous method, the number of samples is a random variable whose statistics are specified by the weights. Since the estimates $\tilde{I}_{i,N_i}^*$ are independent and approximately Normal random variables with mean I_i and variance σ_i^2/N_i for large $N_1, \ldots, N_M$, the variance of the estimate $\tilde{I}_N^*$ is given by

$$\mathrm{Var}\left[\tilde{I}_N^*\right] = \sum_{i=1}^{M} w_i^2 \frac{\sigma_i^2}{N_i} \tag{19}$$

Given the values of the weights so that $f(\boldsymbol{\theta})$ is specified, the optimal values of N_i, $i = 1, \ldots, M$, are those which minimize the $\mathrm{Var}\left[\tilde{I}_N^*\right]$ in (19) subject to the constraint $\sum_{i=1}^{M} N_i = N$. The minimization yields

$$N_i = \frac{w_i \sigma_i}{\sum_{j=1}^{M} w_j \sigma_j} N \tag{20}$$

with the resulting variance given by

$$\mathrm{Var}\left[\tilde{I}_N^*\right] = \frac{\bar{\sigma}^2}{N} \tag{21}$$

Note that each standard deviation σ_i is usually unknown before the simulation process is begun. Although one can estimate it with a few samples in a startup procedure, the error in the estimate of the variance may distort the optimality and hence this choice is not suggested. It is interesting to note that these results are analogous to those derived for stratified sampling [6].

An alternative choice is to take $N_i = w_i N$. Substituting $N_i = w_i N$ into (19), the variance of $\tilde{I}_N^*$ under this choice of N_i is given by

$$\mathrm{Var}\left[\tilde{I}_N^*\right] = \frac{1}{N} \sum_{i=1}^{M} w_i \sigma_i^2 = \frac{1}{N}\left[\bar{\sigma}^2 + \sum_{i=1}^{M} w_i (\sigma_i - \bar{\sigma})^2\right] \tag{22}$$

which is smaller than $\mathrm{Var}[\tilde{I}_N]$ given in (15). Such reduction of variance is due to the deterministic nature of the number of samples N_i used for estimating the integrals I_i. The variances of

$\tilde{I}_N$ and $\tilde{I}_N^*$ coincide in the special case when all the I_is are equal. Note that the component $\sum_{i=1}^{M} w_i(\sigma_i - \bar{\sigma}_i)^2/N$ in (22) and (15) is due to the difference in the variance σ_i of I_i among different design points, and it can be eliminated by using N_i given by (20).

3.1. Choice of weights

For given sample size N, the number of samples generated around the ith design point is proportional to the associated weight w_i. Thus, when choosing the value of w_i, one should take into consideration the relative importance of the ith design point in contributing to the value of the reliability integral during the simulation process.

One reasonable approach is to choose the weights proportional to the contribution to the reliability integral of the integral over the neighborhood of the ith design point. Although the integral around the ith design point is unknown before the sampling process, it can be estimated approximately using the asymptotic contribution, $\tilde{I}_i$, of the ith design point to the reliability integral. Thus, the weights can be chosen in the form

$$w_i = \frac{\hat{I}_i}{\sum_{j=1}^{M} \hat{I}_j}, \; i = 1, \ldots, M \tag{23}$$

where each $\hat{I}_i$, $i = 1, \ldots, M$, is given by (4). This choice is the same as the one proposed in [11] for the classical reliability integral (1).

Another reasonable approach is to choose the weights to be proportional to the value of the integrand $F(\boldsymbol{\theta})p(\boldsymbol{\theta})$ of the original reliability integral evaluated at the design point $\boldsymbol{\theta}_i^*$. Using the asymptotic result (4), the weights can be written in the form

$$w_i = \frac{F(\boldsymbol{\theta}_i^*)p(\boldsymbol{\theta}_i^*)}{\sum_{j=1}^{M} F(\boldsymbol{\theta}_j^*)p(\boldsymbol{\theta}_j^*)} = \frac{\hat{I}_i\sqrt{|\mathbf{H}(\boldsymbol{\theta}_i^*)|}}{\sum_{j=1}^{M} \hat{I}_j\sqrt{|\mathbf{H}(\boldsymbol{\theta}_j^*)|}}, \quad i = 1, \ldots, M \tag{24}$$

where $\hat{I}_i$ is the asymptotic contribution to the value of the reliability integral from the ith design point, and $\sqrt{|\mathbf{H}(\boldsymbol{\theta}_i^*)|}$ accounts for the curvature of the integrand function $F(\boldsymbol{\theta})p(\boldsymbol{\theta})$ evaluated at the ith design point. This choice applied to the reliability integral (2) is similar to the choice proposed in [7] for the importance sampling technique to account for the multiple design points encountered in the classical reliability integral (1).

Ideally, the optimal values of the weights should be selected as those which minimize the variance $\sigma^2(f)/N$, where $\sigma^2(f)$ is given in (14). Using (13), the variance σ^2 in (14) takes the form:

$$\sigma^2(f) = \sum_{i=1}^{M} w_i \int_{\Theta} \left[\frac{F(\boldsymbol{\theta})p(\boldsymbol{\theta})}{f(\boldsymbol{\theta})}\right]^2 G_i(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} - I^2 \tag{25}$$

An approximate expression for the optimal weights is derived next which sheds lights on how the weights could be chosen properly. Since the second term in (25) is constant, minimizing σ^2 leads to minimizing the first sum as a function of the weights. In the following, let the probability density function G_i be Normal with mean $\boldsymbol{\theta}_i^*$ and covariance matrix $\mathbf{C}_i$. Note that for the importance sampling distribution $f(\boldsymbol{\theta})$ corresponding to the optimal choice of weights, the quotient Fp/f would be relatively flat in the neighborhood of the design points $\boldsymbol{\theta}_i^*$, $i = 1, \ldots, M$, when compared to G_i which is peaked at the design point $\boldsymbol{\theta}_i^*$. For well-separated design points, the main contribution to the ith integral within the sum in (25) then comes from the integration over the neighborhood of the ith design point, where $f(\boldsymbol{\theta}) \approx w_i G_i(\boldsymbol{\theta})$. Thus, approximating the ith integral in the sum of (25) with an integral over the neighborhood $\mathbb{D}_i$ of $\boldsymbol{\theta}_i^*$ and replacing f in the denominator of the resulting integral with $w_i G_i$, we have

$$\int_{\Theta} \left[\frac{F(\boldsymbol{\theta})p(\boldsymbol{\theta})}{f(\boldsymbol{\theta})}\right]^2 G_i(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} \approx \frac{1}{w_i^2}\int_{\mathbb{D}_i} \frac{F(\boldsymbol{\theta})^2 p(\boldsymbol{\theta})^2}{G_i(\boldsymbol{\theta})}\mathrm{d}\boldsymbol{\theta} \tag{26}$$

Applying the approximation (4) to the integral in (26) with design point $\boldsymbol{\theta}_i^*$, it can be readily shown that

$$\int_{\Theta} \left[\frac{F(\boldsymbol{\theta})p(\boldsymbol{\theta})}{f(\boldsymbol{\theta})}\right]^2 G_i(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} \approx \frac{\hat{I}_i^2 \gamma_i^2}{w_i^2} \tag{27}$$

where $\hat{I}_i$ is the asymptotic contribution to the reliability integral I, given by (4), and

$$\gamma_i^2 = \frac{\left|\mathbf{H}(\boldsymbol{\theta}_i^*)\right|\sqrt{|\mathbf{C}_i|}}{\sqrt{\left|2\mathbf{H}(\boldsymbol{\theta}_i^*) - \mathbf{C}_i^{-1}\right|}} \tag{28}$$

in which $\mathbf{H}(\boldsymbol{\theta}_i^*)$ is the same Hessian matrix as the one used in (4). The approximation in (27) is valid for the matrix $(2\mathbf{H}(\boldsymbol{\theta}_i^*) - \mathbf{C}_i^{-1})$ being positive definite so that $\boldsymbol{\theta}_i^*$ is a design point of the integrand in (26).

Substituting (27) into (25) and minimizing the resulting expression for $\sigma^2(f)$ with respect to w_i, $i = 1, \ldots, M$, subject to the constraints $\sum_{i=1}^{M} w_i = 1$, yields the following approximation for the optimal weights:

$$w_{i,\mathrm{opt}} \approx \frac{\hat{I}_i \gamma_i}{\sum_{j=1}^{M} \hat{I}_j \gamma_j}, \; i = 1, \ldots, M \tag{29}$$

Eq. (29) suggests the optimal weights be chosen proportional to the product of the asymptotic contribution from the ith design point and the parameter γ_i. The first factor accounts for the fact that design points having higher asymptotic contributions to the reliability integral should be given higher weights, and hence higher number of samples generated in their neighborhood. To understand the significance of the second factor, γ_i, consider two design points $\boldsymbol{\theta}_i^*$ and $\boldsymbol{\theta}_j^*$ having exactly the same asymptotic contribution $\hat{I}_i = \hat{I}_j$, and assume that G_i and G_j have the same covariance matrix $\mathbf{C}$. The number of samples $N_i = w_i N$ generated around each design point using the Normal distribution with the same covariance matrix $\mathbf{C}$ depends also on the curvature of Fp, given by $\mathbf{H}(\boldsymbol{\theta}^*)$ at the design point. The higher the curvature of Fp, the more peaked the function Fp is as compared to the importance sampling distribution $G_i(\boldsymbol{\theta})$ around the design point, and therefore, the more the number of samples needed to get a good importance sampling estimate of the integral over the region corresponding to the design point. In the limiting case for which $\mathbf{H}(\boldsymbol{\theta}_i^*) = \mathbf{C}_i^{-1}$, (29) gives $\gamma_i = 1$ which implies that the choice of weight for the ith design point is independent of γ_i. In fact, had the function Fp been exactly represented by a Normal distribution centered at $\boldsymbol{\theta}_i^*$, only one sample would be sufficient to give the exact value.

Note that when $\mathbf{H}(\boldsymbol{\theta}_i^*) = \mathbf{C}_i^{-1}$, $i = 1, \ldots, M$, (28) gives $\gamma_i = 1$ for all i, and hence the approximate optimal choice of weights by (29) coincides with the choice in (23) in which the weights are proportional to the asymptotic contributions.

4. Applications

The accuracy and efficiency of both the asymptotic approximation and the importance sampling methodology for estimating reliability integrals of the form (2) with multiple design points are investigated by computing the failure probabilities of a passively-damped structure subjected to a stationary zero-mean Gaussian white-noise base excitation. The first example considers a two-degree-of-freedom system which helps to graphically illustrate the design points and their contributions to the reliability integrals. The second example considers a 10-DOF shear building which presents a problem of practical interest as numerical integration becomes prohibitive because of the high dimension of the reliability integral encountered.

In both examples, for a given $\boldsymbol{\theta}$, failure is assumed to occur when the stationary portion of a response quantity $r(t; \boldsymbol{\theta})$ exceeds some critical level b over a duration T. For a high threshold level b, it can be assumed that the events of crossing such a level are independent, in which case the conditional failure probability $F(\boldsymbol{\theta})$ is approximated, using results from random vibration theory [23], by

$$F(\boldsymbol{\theta}) \approx 1 - \exp[-2\nu(\boldsymbol{\theta})T] \tag{30}$$

where, for zero-mean Gaussian processes, the expected rate of upcrossing $\nu(\boldsymbol{\theta})$ through level b for a given $\boldsymbol{\theta}$ is

$$\nu(\boldsymbol{\theta}) = \frac{\sigma_{\dot{r}}(\boldsymbol{\theta})}{2\pi\sigma_r(\boldsymbol{\theta})} \exp\left[\frac{-b^2}{2\sigma_r^2(\boldsymbol{\theta})}\right] \tag{31}$$

and $\sigma_r(\boldsymbol{\theta})$ and $\sigma_{\dot{r}}(\boldsymbol{\theta})$ are, respectively, the conditional standard deviation of response $r(t; \boldsymbol{\theta})$ and its time derivative for a given $\boldsymbol{\theta}$. Using the Theorem of Total Probability, the unconditional failure probability I of the system is given by the integral (2).

4.1. Two-DOF system

Consider a two degree-of-freedom (DOF) system consisting of a structure with a tuned mass damper (TMD) specified by the following parameters (see Fig. 1): mass of structure m_0, natural frequency of structure $\omega_0 = \sqrt{k_0/m_0}$, structural damping ratio $\zeta_0 = c_0/2\sqrt{k_0 m_0}$, mass ratio $\mu = m_1/m_0$, fixed-base natural frequency of TMD $\omega_1 = \sqrt{k_1/m_1}$ and TMD damping ratio $\zeta_1 = c_1/2\sqrt{k_1 m_1}$. The stiffness k_0 and damping ratio ζ_0 of the structure are assumed to be uncertain. They are parameterized by $k_0 = \hat{k}_0\theta_1$ and $\zeta_0 = \hat{\zeta}_0\theta_2$, where $\hat{k}_0$ and $\hat{\zeta}_0$ are the most probable value of k_0 and ζ_0, respectively, and θ_1 and θ_2 are dimensionless quantities representing the uncertain parameters of the system. The uncertainties in k_0 and ζ_0 are then quantified by choosing θ_1 and θ_2 to be independent and lognormally distributed with most probable value (MPV) $\hat{\theta}_1 = \hat{\theta}_2 = 1$, and standard deviation γ_1 and γ_2, respectively. Equivalently, γ_1 and γ_2 measure the level of uncertainties of k_0 and ζ_0, respectively. The other parameters of the system are assumed to be deterministic. The following values for the system parameters are assumed: $m_0 = 1 \times 10^5$ kg, $\hat{\zeta}_0 = 1\%$, $\mu = 1\%$, $\zeta_1 = 1\%$ and $\omega_1 = 0.8\ \hat{\omega}_0$, where $\hat{\omega}_0 = \sqrt{\hat{k}_0/m_0} = 5\pi\,\text{rad/s}$ is the MPV of the frequency of the structure corresponding to the MPV of k_0.

The response quantity of interest $r(t;\boldsymbol{\theta})$ is the displacement of the structure relative to the ground. The threshold value b in (31) is assumed to be four times the standard deviation of the displacement $r(t;\boldsymbol{\theta})$ of the nominal structure in the absence of the TMD, i.e. when $\theta = 1$ and $\mu = 0$. The duration T is taken to be 10 times the natural period of the nominal structure in the absence of the TMD. For this example, only the asymptotic approximation for the system failure probability I is computed and compared with the "exact" value obtained by numerical integration.

In all the cases considered herein, the level of uncertainty of ζ_0 is fixed at $\gamma_2 = 0.3$. Results are presented for three levels of uncertainty of k_0, namely $\gamma_1 = 0.25$, 0.4 and 0.5, which are designated, respectively, by Case 1, 2 and 3. It is found that for all of these three cases there exist two design points. To gain insight into the contribution of the integral over the neighborhood of the design points for different levels of uncertainty γ_1, the variation of the integrand function

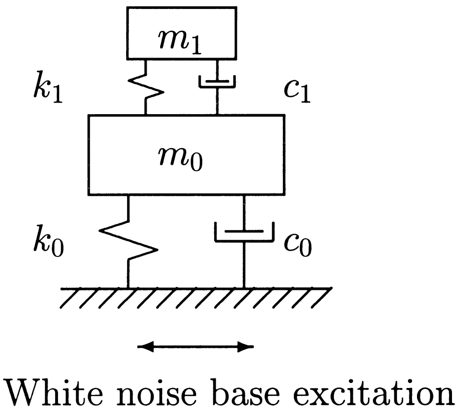
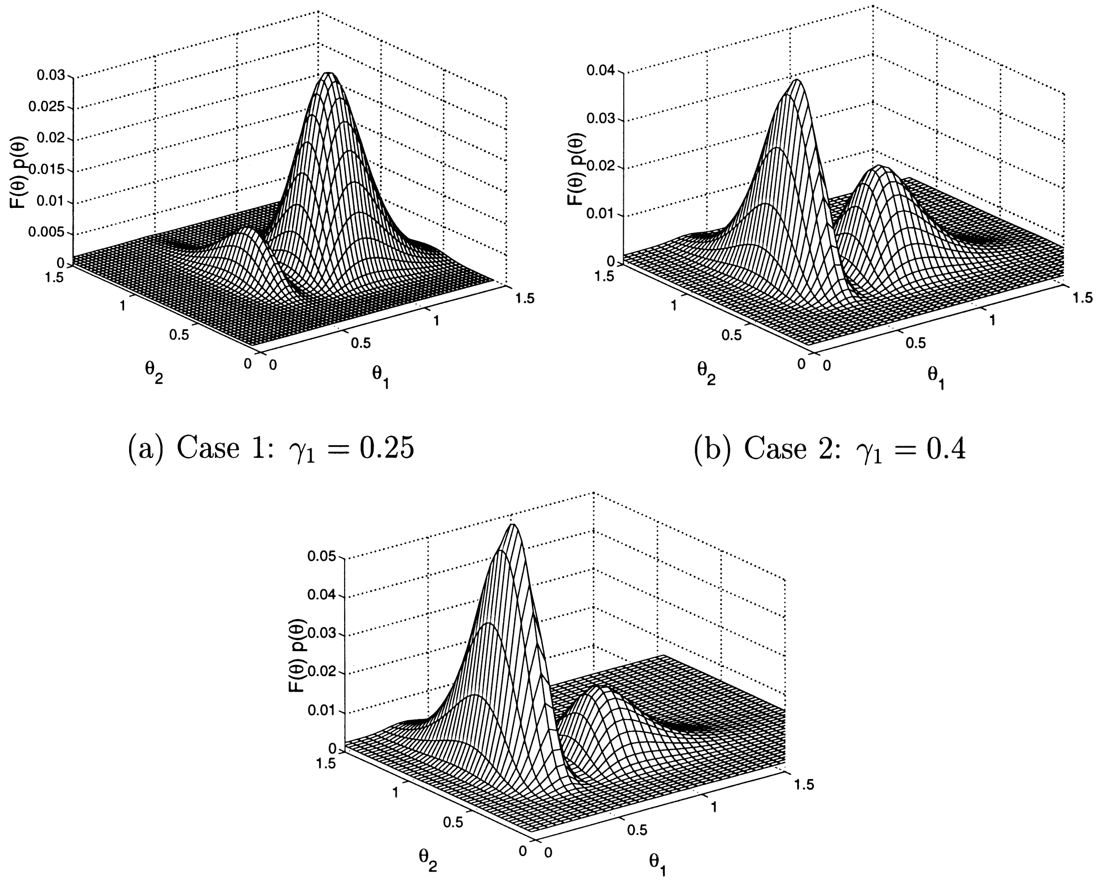


Fig. 1. Two-DOF system.

$F(\boldsymbol{\theta})p(\boldsymbol{\theta})$ is plotted in Fig. 2(a)–(c) for Cases 1, 2 and 3, respectively. For illustration purposes, the design point near the MPV, $\hat{\boldsymbol{\theta}} = (1, 1)$, is designated as design point 1 and denoted by $\boldsymbol{\theta}_1^*$, while the other design point farther away from the MPV is designated as design point 2 and denoted by $\boldsymbol{\theta}_2^*$. Table 1 shows the values of the design points $\boldsymbol{\theta}_1^*$ and $\boldsymbol{\theta}_2^*$ for Cases 1, 2 and 3. Each column in the table contains the two components of each design point in the parameter space $\boldsymbol{\theta}$. It is noted that the difference in the first component between $\boldsymbol{\theta}_1^*$ and $\boldsymbol{\theta}_2^*$ is larger than that for the second component which is indicative of the fact that the failure probability is more sensitive to uncertainty in the stiffness parameter than it is to the damping ratio.

To get insight into the relative contribution of the design points to the failure probability I for different levels of uncertainty, the fractional asymptotic contribution of a design point $\boldsymbol{\theta}_i^*$, defined here as $\beta_i = \hat{I}_i/\hat{I}$, is computed. The fractional asymptotic contribution reflects the relative contribution of the integral from the neighborhood of the design point and hence the importance of the design point in reliability computations. The fractional asymptotic contribution β_i, $i = 1, 2$,



(a) Case 1: $\gamma_1 = 0.25$

(b) Case 2: $\gamma_1 = 0.4$

(c) Case 3: $\gamma_1 = 0.5$

Fig. 2. Integrand $F(\boldsymbol{\theta})p(\boldsymbol{\theta})$ for Cases 1, 2 and 3.

and the total asymptotic approximation $\hat{I} = \hat{I}_1 + \hat{I}_2$ are shown in Table 2 for the three cases. Column 6 in Table 2 reports the results obtained by transforming the random variables θ_1 and θ_2 to standard Normal ones and then applying the asymptotic approximation to the transformed integral over the standard Normal space. For comparison purposes, the 'exact' values of the failure probabilities based on numerical integration are also reported in this table.

It is seen from Fig. 2 and Table 2 that as γ_1 increases from 0.25 to 0.5, β_1 decreases from 90 to 31%, while β_2 increases from 10 to 69%. This means that as the level of uncertainty in the stiffness parameter θ_1 increases, the contribution of the integral over the neighborhood of design point 1 to the total probability of failure reduces, and hence design point 1 is expected to be less important in accounting for the total failure probability. On the other hand, as the level of uncertainty increases, the contribution of the integral around the neighborhood of the design point 2 is important in obtaining more accurate estimates of the failure probability. The asymptotic estimate in the original space of random variables is a very good approximation to the value of the reliability integral, provided that both design points are used. It can be deduced from Table 2 that the estimate from one design point only can be inaccurate, especially if the single design point found from an optimization algorithm is the one corresponding to the smaller β_i. Finally, comparing columns 5 and 6 in Table 2, it can be seen that depending on the value of γ_1, the approximation in the transformed standard Normal space may give worse or better estimates than the approximation in the original space.

4.2. 10-DOF shear building with TMD

Consider a 10-story shear building equipped with a tuned mass damper at the roof. The building is modeled by a 10-DOF spring-mass-damper system and the TMD is modeled by a SDOF mass-spring-damper attached to the 10th DOF of the building, as shown in Fig. 3. The lumped mass of all stories are $m_i = 1 \times 10^5$ kg, $i = 1, \ldots, 10$. The interstory stiffness k_i of all the stories are assumed to be uncertain and they are parameterized by $k_i = \hat{k}_i\theta_i$, $i = 1, \ldots, 10$, where

Table 1
Design points of study cases for 2 DOF system

Case 1		Case 2		Case 3	
$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$	$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$	$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$
0.935	0.551	0.917	0.510	0.912	0.492
0.681	0.793	0.684	0.817	0.684	0.828

Table 2
Asymptotic results for 2 DOF system

Case	γ_1	β_1	β_2	$\hat{I}$ (original)	$\hat{I}$ (normal)	I (integration)
1	0.25	0.90	0.10	4.85×10^{-3}	5.09×10^{-3}	5.02×10^{-3}
2	0.40	0.48	0.52	6.75×10^{-3}	6.90×10^{-3}	7.21×10^{-3}
3	0.50	0.31	0.69	9.00×10^{-3}	8.93×10^{-3}	9.29×10^{-3}

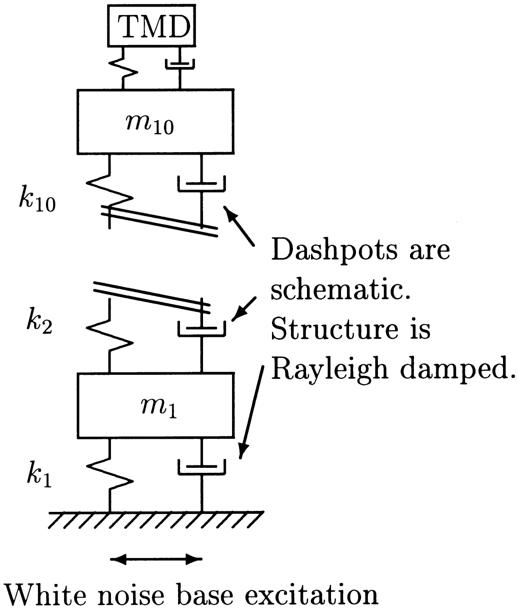


Fig. 3. 10-DOF shear building model with TMD.

each $\hat{k}_i = 180 \times 10^6$ N/m, is the most probable value of k_i and the θ_is forming the vector $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_{10}]^T$ are nondimensional uncertain parameters modeled by random variables. The nominal model is defined here as the 10-DOF building model with parameters $\theta_i = 1$ for all i when the TMD is not installed. The fundamental frequency of the nominal model is computed to be about 1 Hz. To account for the uncertainty of the interstory stiffnesses as well as their statistical correlation, the uncertain parameters θ_i are assumed to be Normal with mean $\hat{\theta}_i = 1$ and correlation structure described by the exponential decay law

$$\mathrm{E}\Big[(\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j)\Big] = \gamma^2 \exp\big[-(j-i)^2/\lambda^2\big], \quad i,j = 1, \ldots, 10 \tag{32}$$

where γ is the standard deviation of each component θ_i in the random vector $\boldsymbol{\theta}$, and λ is a characteristic story-correlation number. It can be readily seen that γ is the coefficient of variation of k_i, $i = 1, \ldots, 10$. Chosen values for γ will be small so that the probability that any θ_i is negative will be negligible. The correlation number λ is chosen to be $\lambda = 3$ which roughly implies a significant correlation between interstory stiffnesses within 3 stories apart.

The 10-DOF building is assumed to be Rayleigh damped with damping matrix $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}(\boldsymbol{\theta})$, where $\mathbf{M}$ is the mass matrix and $\mathbf{K}(\boldsymbol{\theta})$ is the stiffness matrix of the 10-DOF building for a given $\boldsymbol{\theta}$; α and β are the Rayleigh damping parameters and are assumed to be such that the nominal building has 1% modal damping in the first and second modes of vibration. The mass of the TMD is 1% of the total mass of the building, and the stiffness of the TMD is such that the fixed-base natural frequency of the TMD is 0.8 of the first mode natural frequency of the nominal building, representing a nearly-tuned condition. The fixed-base TMD is assumed to have 1% of critical damping. The response quantity of interest, $r(t; \boldsymbol{\theta})$, is the roof displacement relative to the ground.

The reliability computations are carried out in the transformed space of independent standard Normal random variables. For this, the set of correlated random variables $\boldsymbol{\theta}$ are transformed to a set of independent standard Normal random variables $\boldsymbol{\eta} = [\eta_1, \dots, \eta_{10}]^T$. Asymptotic and importance sampling techniques for evaluating I are then applied to the transformed integral with respect to $\boldsymbol{\eta}$ over the 10-dimensional space of $\boldsymbol{\eta}$.

4.2.1. Asymptotic approximation

Three cases are considered, designated as Case 1, 2 and 3, which correspond to different levels of uncertainty in the interstory stiffnesses with coefficient of variation $\gamma = 0.2$, 0.25 and 0.3, respectively. It is found that two design points, designated by $\boldsymbol{\theta}_1^*$ and $\boldsymbol{\theta}_2^*$, exist in all of the three cases. The corresponding design points in the transformed parameter space are denoted by $\boldsymbol{\eta}_1^*$ and $\boldsymbol{\eta}_2^*$. The design points for the three cases in the original parameter space $\boldsymbol{\Theta}$ are tabulated in Table 3. Each column in the table shows the 10 components of the design point. Observe that in all of the cases only the first two components of $\boldsymbol{\theta}_1^*$ are significantly different from their most probable values, while the first four or five components of $\boldsymbol{\theta}_2^*$ are found to be significantly different from their most probable values. Those components which are significantly different from their most probable values are usually the sensitive parameters of the uncertain system. It is interesting to note that in all three cases, the design points are approximately the same for different levels of uncertainty considered, especially for the second design point $\boldsymbol{\theta}_2^*$.

Given the design points, the asymptotic approximation $\hat{I}$ to I can readily be computed as $\hat{I} = \hat{I}_1 + \hat{I}_2$, where $\hat{I}_1$, and $\hat{I}_2$ are the asymptotic contributions from $\boldsymbol{\theta}_1^*$ and $\boldsymbol{\theta}_2^*$ according to (4). The fractional asymptotic contribution, $\beta_i = \hat{I}_i/\hat{I}$, of each design point and the total asymptotic approximation $\hat{I}$ are tabulated in Table 4. Note that as γ increases from 0.2 to 0.3, β_1 decreases from 0.71 in Case 1 to 0.37 in Case 3, while β_2 increases from 0.29 to 0.63. Assuming that the exact values of failure probabilities follow a similar trend, this means that as the level of uncertainty in the interstory stiffnesses increases, the contribution of the integral over the neighborhood of $\boldsymbol{\theta}_1^*$ to the value of I reduces, and hence $\boldsymbol{\theta}_1^*$ is expected to be less important in the reliability computations. On the other hand, as the level of uncertainty increases, $\boldsymbol{\theta}_2^*$ is more important in obtaining a more accurate estimate of failure probability. Coupling this with the fact

Table 3
Design points for 10-DOF shear building

Case 1		Case 2		Case 3	
$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$	$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$	$\boldsymbol{\theta}_1^*$	$\boldsymbol{\theta}_2^*$
0.67	0.39	0.59	0.38	0.54	0.37
0.79	0.40	0.75	0.39	0.73	0.38
0.95	0.51	0.95	0.50	0.97	0.50
1.07	0.65	1.11	0.65	1.15	0.65
1.13	0.78	1.18	0.79	1.22	0.79
1.12	0.87	1.16	0.88	1.20	0.88
1.09	0.93	1.11	0.93	1.14	0.93
1.05	0.95	1.06	0.96	1.07	0.96
1.01	0.97	1.01	0.97	1.02	0.97
0.99	0.98	0.99	0.98	0.98	0.98

Table 4
Study cases and results for 10-DOF shear building

Case	γ	β_1	β_2	$\hat{I}$
1	0.20	0.71	0.29	4.77×10^{-3}
2	0.25	0.47	0.53	1.35×10^{-2}
3	0.30	0.37	0.63	2.82×10^{-2}

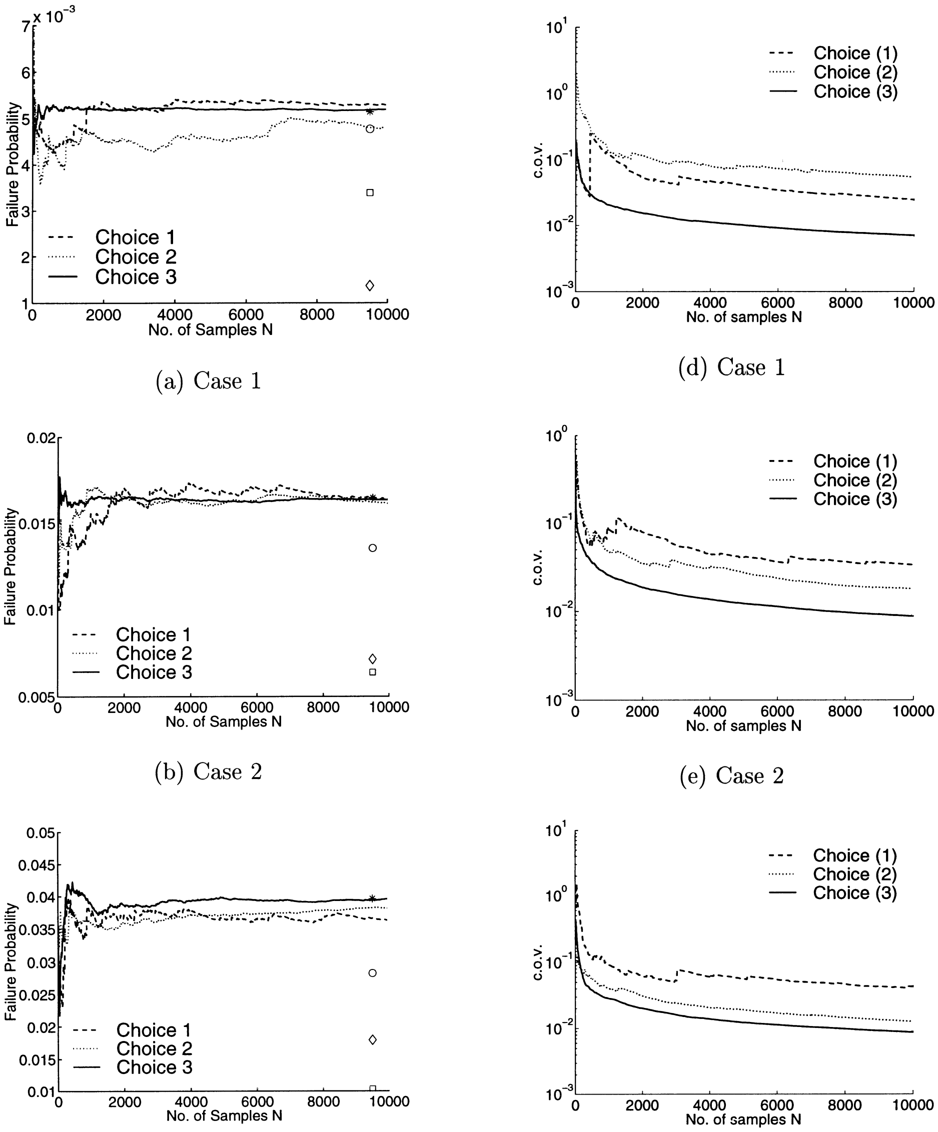
that $\boldsymbol{\theta}_2^*$ is far away from the most probable value and hence has small plausibility, one sees the intuitive fact that as the level of uncertainty increases, models with small plausibilities may assume great importance in reliability calculations. These conclusions are similar to those in the previous example.

4.2.2. Importance sampling simulation

The accuracy and convergence of the importance sampling technique proposed in this study is explored. To investigate the importance of utilizing information from both of the design points in the importance sampling process, simulations are carried out separately for the following three choices of sampling distribution given in (13): (1) $w_1 = 1$ and $w_2 = 0$, (2) $w_1 = 0$ and $w_2 = 1$, and (3) $w_1 = \beta_1 = \hat{I}_1/(\hat{I}_1 + \hat{I}_2)$ and $w_2 = \beta_2 = \hat{I}_2/(\hat{I}_1 + \hat{I}_2)$. Choices (1) and (2) correspond to the choice of sampling distribution when only one design point, $\boldsymbol{\eta}_1^*$ for choice (1) and $\boldsymbol{\eta}_2^*$ for choice (2), is found. Choice (3) is the proposed sampling distribution that utilizes information from both design points. The number of samples N_i for the ith design point is chosen to be $N_i = w_i N$, $i = 1, 2$. The functions $G_i(\boldsymbol{\eta})$ for (13) are chosen to be Normally distributed, with most probable value equal to $\boldsymbol{\eta}_i^*$ corresponding to the ith design point and covariance matrix equal $\mathbf{C}_i$ to the identity matrix $\mathbf{I}$, that is, the covariance matrix corresponding to that of the original distribution after transformation to the $\boldsymbol{\eta}$-space. It is worth noting that the choice $\mathbf{C}_i = \mathbf{H}^{-1}(\boldsymbol{\theta}_i^*)$ was not used because it does not satisfy the conditions for getting a bounded variance of the estimate for all numerical cases considered in this example.

Fig. 4(a)–(c) shows the importance sampling estimates as a function of number of samples for Cases 1, 2 and 3, respectively. The corresponding coefficients of variation, $\mathrm{cov}[\tilde{I}_N] = \tilde{\sigma}_N/\tilde{I}_N$ of the estimate $\tilde{I}_N$, where $\tilde{\sigma}_N$ is computed based on (12), are respectively plotted in Fig. 4(d)–(f). The coefficient of variation is often used to assess the error in simulation results and provides guidance in terminating the simulation process once the error is below a specified threshold. Results using up to 10,000 samples are shown in the figures. The dashed, dotted and solid lines correspond to sampling histories for choices (1), (2) and (3), respectively. For comparison purposes, the asymptotic contributions $\hat{I}_1$, $\hat{I}_2$ and $\hat{I} = \hat{I}_1 + \hat{I}_2$ are also marked in these figures with a square, diamond and circle, respectively. Exact solutions for the reliability integrals are not feasible in this example due to the large dimension of the integrals. Simulation results from all the three choices of sampling distribution using 100,000 samples, however, show that they all practically converge to the same value. For discussion purposes, such a value can be taken as the exact solution and is marked with star in the figure for each case.

From Fig. 4, it is seen that the asymptotic approximation $\hat{I}$ is a good approximation to the failure probability I, while asymptotic approximations using only one design point, $\hat{I}_1$ or $\hat{I}_2$, may not be necessarily close to I, depending on the level of uncertainty γ. From the sampling histories



(a) Case 1 (d) Case 1

(b) Case 2 (e) Case 2

(c) Case 3 (f) Case 3

Fig. 4. Simulation histories for Cases 1, 2 and 3. □ $\hat{I}_1$; ◇ $\hat{I}_2$; ○ $\hat{I}$; * Exact.

corresponding to the different choices of sampling distribution, it is observed that simulation results using the sampling distribution peaked at a design point having high asymptotic contribution tend to have less variance. This can also be inferred from the coefficients of variation. Fig. 4 also shows that the coefficient of variation for Choice (3) is always smaller than those for Choices (1) and (2). Also, the largest coefficient of variation among Choices (1) and (2) is approximately an order of magnitude greater than that of Choice (3). It should be noted that conventional optimization methods used to search for a design point may yield a local minimum corresponding to the design point with the least contribution to the asymptotic estimate of the reliability integral. The importance sampling simulation based on this design point would lead to large variance and, hence, would lose its efficiency. In all the cases, the importance sampling estimates using Choice (3) show faster convergence than those using Choice (1) or (2), demonstrating the efficiency of the proposed importance sampling methodology.

Note that the importance sampling estimates using Choice (1) or Choice (2) could be biased when N is small. For example, in Case 2, when N is about 1000, the estimate by Choice (1) is significantly smaller than the exact value. Such difference, however, is not reflected in the estimated coefficient of variation. Indeed, the initial portion of the estimated coefficient of variation for N less than 1000 is quite small which could lead to the erroneous conclusion that the importance sampling estimate has converged to within 6%. In this case, where N is not large enough, the estimated coefficient of variation is much smaller than the actual one, and the former can no longer give a faithful indication of the accuracy of the importance sampling estimate [10,24].

Bias is more likely to occur for smaller sampling size N when design points distant from the center of the sampling distribution and of significant contribution are not included in the sampling distribution. In such a case, samples generated according to the sampling distribution which excludes the design points of significant contribution tend to cluster around the center of the sampling distribution. The chance of having samples generated in the neighborhood of the excluded design points is small when N is small, and thus the contributions of the excluded design points to the reliability integral are not reflected in the simulation, thereby causing the bias in the importance sampling estimate.

For sufficiently large N, however, there could be a small number of samples generated in the neighborhood of the excluded design points which may result in sudden 'jumps' in the plots of the estimate, as can be observed in the sampling histories and coefficient of variation plots for Choices (1) and (2) in Fig. 4. Such jumps occur because the quotient $F(\boldsymbol{\theta})p(\boldsymbol{\theta})/f(\boldsymbol{\theta})$ has a very high value in the neighborhood of an excluded design point, as the denominator, $f(\boldsymbol{\theta})$, has negligible value there while the numerator, $F(\boldsymbol{\theta})p(\boldsymbol{\theta})$, takes on significant values in the neighborhood of the design point. The existence of jumps in a plot of the estimates for increasing sample size may indicate the existence of design points of high contributions which are excluded in the sampling distribution.

When an importance sampling method is applied using only one design point, a large variance in the sampling history and a significant difference in the importance sampling estimate $\tilde{I}_N$ from the asymptotic approximation can be a good indication of the existence of other design points of high contribution to the reliability integral. In the case where large variance is observed using the first design point found, searching for the other design points may be worthwhile, as can be seen in Fig. 4 by the significant improvement in the convergence behavior of the importance sampling technique when information from both design points is used to choose the importance sampling density.

The effect of the choice of weights w_i on the efficiency of the proposed importance sampling estimate is investigated next. A parametric study on the variance σ^2 of $\kappa(\boldsymbol{\theta}) = F(\boldsymbol{\theta})p(\boldsymbol{\theta})/f(\boldsymbol{\theta})$ is performed with respect to the weight w_1 associated with the first design point, while $w_2 = 1 - w_1$. The variance σ^2 directly affects the variance of the importance sampling estimate using Choice (3) since the variance of the latter is σ^2/N. The values of σ^2 have been estimated by importance sampling simulation using Choice (3) with $N = 10,000$ samples. For comparison purposes, the ratio of the variance σ^2 for a given w_1 to the smallest variance σ_0^2 attained at the optimal value of w_1 are plotted in Fig. 5 for Cases 1, 2 and 3. Note that the cases $w_1 = 0$ and $w_1 = 1$ correspond to simulations using Choice (2) and Choice (1), respectively. From these figures, it is seen that the variance ratio could be orders of magnitude greater than 1 when $w_1 = 0$ [Choice (2)] or $w_1 = 1$ [Choice (1)]. For example, in Case 1, when $w_1 = 0$, $\sigma^2/\sigma_0^2 \approx 50$. This means on average it takes about 50 times more samples for Choice (2) than Choice (3) with $w_1 = 0.6$, $w_2 = 0.4$ to achieve the same variance in the importance sampling estimate.

The variation of the variance ratio shown in these figures demonstrates a convex trend and hence that an optimal choice of the weights is possible to minimize the variance. The values of w_1 based on the choices specified by (23), (24) and (29) are also shown in the figures as a square, diamond and circle, respectively. The approximate optimal choice given by (29) and the choice

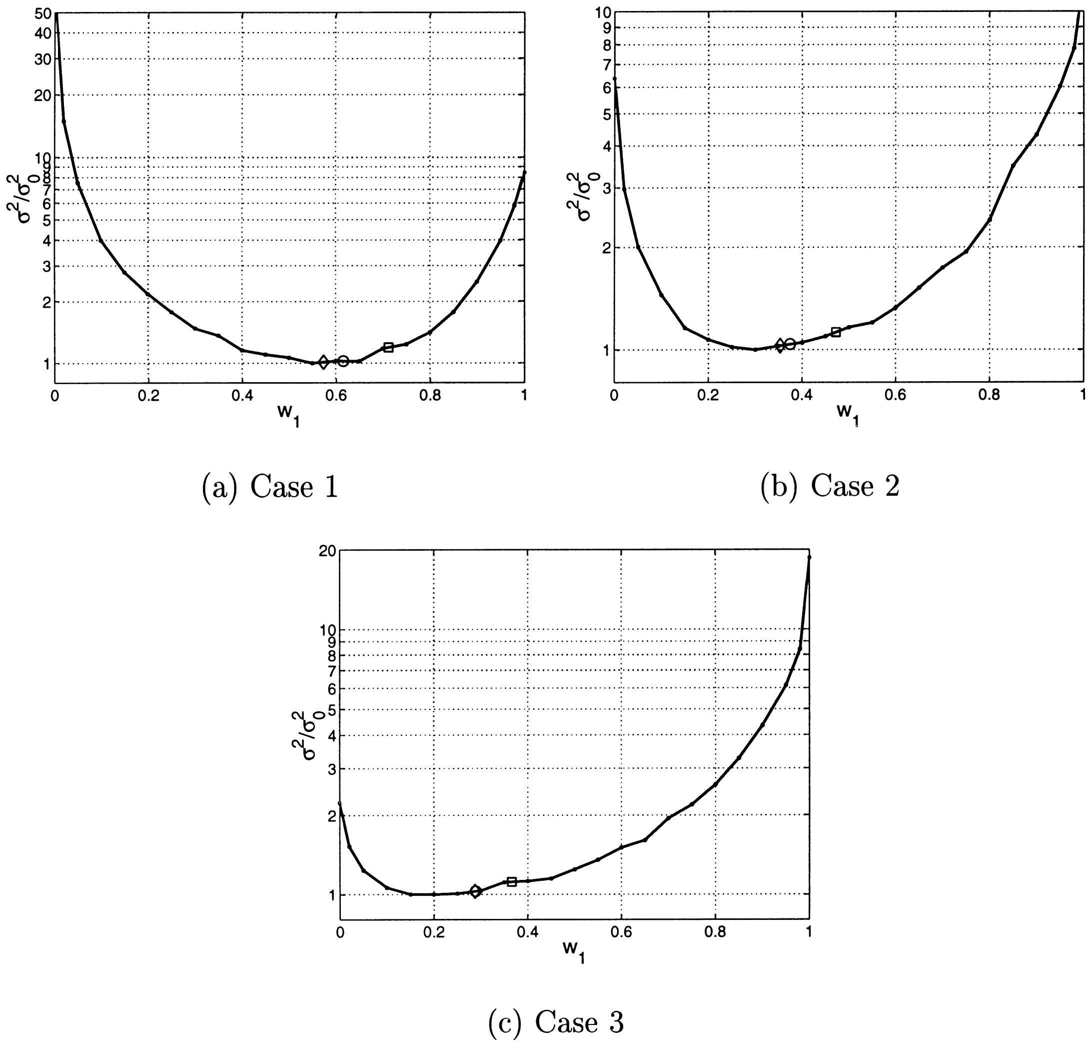


Fig. 5. Variation of variance with weight for Cases 1, 2 and 3. □ Eq. (23); ◇ Eq. (24); ○ Eq. (29).

based on the integrand value at the design points as given by (24) are quite close to the actual optimal value where the variance is minimized. The choice by (23) based on the asymptotic contributions of design points appears to be sub-optimal in these cases. It is noted, however, in the examples considered, the variation of variance is not significant when w_1 varies near the optimal value. This implies that for the cases studied, the importance sampling procedure is almost optimized in terms of the weights if the weights are chosen either according to (23), (24) or (29).

5. Conclusions

Asymptotic expansions and importance sampling techniques have been developed for reliability integrals arising in reliability analysis of uncertain dynamical systems when multiple design points exist. The accuracy of the asymptotic approximation of a reliability integral can be improved to any desired level by generating a sufficiently large number of samples using the importance sampling method. However, bias in the importance sampling estimate can occur for finite sample sizes if all the design points are not properly accounted for in the sampling distribution. The sampling distribution used in the importance sampling procedure presented herein is a multimodal probability distribution which is peaked near the multiple design points. Samples generated according to such a distribution are clustered around the design points which give significant contribution to the reliability integral. The choice of weights used in the sampling distribution has been discussed. An approximate formula for the optimal weights has been derived which expresses the importance of the design points in terms of their asymptotic contribution and the curvature of the integrand at the design point.

Numerical examples on structures equipped with a tuned mass damper, for which two design points are encountered, demonstrate the applicability and efficiency of the proposed techniques. Studies on the contribution of design points to the reliability integrals reflect the fact that as the level of uncertainty increases, models with small plausibilities can assume great importance in reliability calculations, which agrees with intuition.

Acknowledgements

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